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COMMENT

Studies of the spectral dimension for branched Koch curves

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Abstract. The scaling behaviour of Koch curves with a rather general branched structure is studied by means of the exact renormalisation method and the general formula of spectral dimension is obtained. It is proved that the spectral dimension is only dependent on the topology of the curve. The Einstein relation between the random walk dimension, fractal dimension and size-scaling exponent of DC conductivity is also checked.

Recently, there has been increasing interest in systems with a dilatation symmetry or fractal (Mandelbrot 1977, 1982). In general, there are two kinds of fractal: deterministic fractals (self-similar) such as the Koch curve (Gefen *et al* 1983), the Sierpinski gasket (Gefen *et al* 1981, 1984b, Rammal and Toulouse 1982, 1983, Hilfer and Blumen 1984) and the Sierpinski carpet (Gefen *et al* 1984a), and random fractals (statistically self-similar) such as a random walk in free space and the infinite cluster at percolation threshold (Gefen *et al* 1981, Mandelbrot 1982). There are also many disordered systems with dilatation invariance such as linear or branched polymers (Havlin and Ben-Avraham 1982), diffusion-limited aggregates (Witten and Sander 1981, 1983), porous materials (Even *et al* 1984) and others.

Deterministic fractals have become an important and attractive area of study due to the great current theoretical interest in the properties of systems with self-similar structure. Deterministic fractals could be considered as simple models of realistic systems with statistically self-similar structures. For example, the non-branched or branched Koch curves can be related, as a model, to linear or branched polymers.

Stinchcombe (1985) studied the diffusion in a branched Koch curve and reported on the properties of its spectral dimension. Maritan and Stella (1986) studied the spectral dimension for a non-branched Koch curve with long-range interaction. All of them have shown that the Einstein relation

$$\tilde{d}_w = d_f + \tilde{t} \tag{1}$$

between the random walk dimension $\tilde{d}_w$, the fractal dimension d_f and the size-scaling exponent $\tilde{t}$ of the DC conductivity is also valid. The random walk dimension $\tilde{d}_w$ is related to d_f and the spectral dimension d_s as

$$\tilde{d}_w = 2d_f/d_s \tag{2}$$

where d_s is related to the acoustic property. Goldhirsch and Gefen (1986) developed an analytic method for calculating properties of random walks on networks with branches and discussed the comparison with the Kirchhoff rules in the calculation of impedances.

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It is also believed that the spectral dimension is related to the topology of a system and that the same topological systems may have identical spectral dimensions. For example, spectral dimensions are all equal to one for any non-branching Koch curves. Here we will give a simple proof for any branching Koch curves.

In this comment, the renormalisation method has been used to determine scaling exponents for Koch curves with p-branches ($p = 3$-200) at each branched point which has been shown at the first stage in figure 1. The fractal model can be constructed recursively as the limit of a hierarchy of systems, each obtained from the previous one by replacing each bond by a cluster with many bonds as in figure 1. The fractal dimension of the fractal is

$$d_f = \ln\left(2(n+1) + \sum_{i=1}^{p-1} (m_i + 1)\right)(\ln(b))^{-1} \tag{3}$$

where the self-similar scaling unit b is $[2(n+1)+L+1]$ and the meaning of numbers m_i, n and L are shown on figure 1. We study the perpendicular vibrating model where the displacement of points is vertical to the bonds. The spectral dimension is defined as

$$\rho(\omega) \underset{\omega \sim 0}{\sim} \omega^{d_s - 1} \tag{4}$$

where $\rho(\omega)$ is the density of vibrating states. If frequency ω has the scaling behaviour

$$\omega(bR) = b^{-a}\omega(R) \tag{5}$$

where R is the size of system, then the spectral dimension d_s can be calculated from

$$d_s = d_f / a. \tag{6}$$

The exponent a can be calculated exactly by a decimation procedure which has been described in detail by Stinchcombe (1985) for some kinds of branched Koch curves. In our case (see figure 1), we have two kinds of points: p-branching and non-branching (two-branching) points so that two renormalisation equations can be obtained from

$$\tilde{\alpha}_i = \tilde{\alpha}_i(\alpha_2, \alpha_p) \tag{7}$$

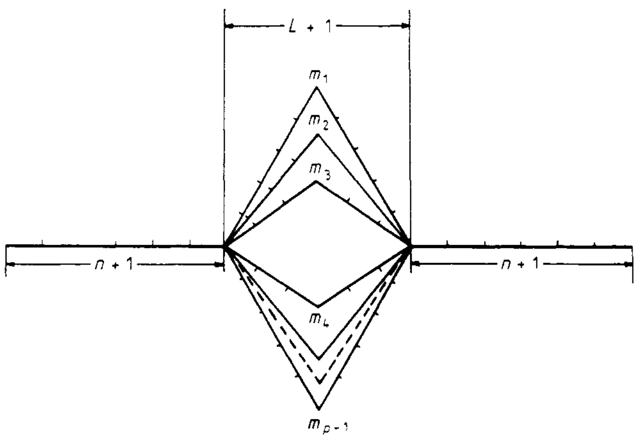

Figure 1. A branched Koch curve with multi-branches at the first stage.

where $\alpha_i = \omega^2/\omega_i^2$, $(i = 2, p)$, $\omega_i^2 = K_i/M_i$, and K_i and M_i are the effective renormalised elastic constant and mass, respectively, of the i-branching point at the nth stage. $\tilde{\alpha}_i$ is the renormalised frequency at the $(n-1)$th stage. The $\tilde{\alpha}_i$ and α_i approach zero as $\omega \to 0$. We do the expansion of $\tilde{\alpha}_i(\alpha_2, \alpha_p)$ near zero (α_2, α_p):

$$\begin{aligned} \tilde{\alpha}_2 &= M_{22}\alpha_2 + M_{2p}\alpha_p \\ \tilde{\alpha}_p &= M_{p2}\alpha_2 + M_{pp}\alpha_p \end{aligned} \tag{8}$$

where $\{M_{ij}\}$ are coefficients which have been obtained in complicated analytical form. Equations (7) can be rewritten into the matrix equation as

$$\tilde{\boldsymbol{\alpha}} = \mathbf{M}\boldsymbol{\alpha}. \tag{9}$$

For a fractal with self-similar structure the stages will be extended to infinity. We have to solve the eigenequation of $\mathbf{M}$

$$\mathbf{M}\boldsymbol{\alpha}^* = \lambda\boldsymbol{\alpha}^* \tag{10}$$

and determine two eigenvalues of $\mathbf{M}$. The largest one $\lambda_{\max}$ corresponds to the fixed point which we need. We have

$$\lambda_{\max} = \tfrac{1}{2}\{M_{22} + M_{pp} + [(M_{22} + M_{pp})^2 - 4(M_{22}M_{pp} - M_{2p}M_{p2})]^{1/2}\}. \tag{11}$$

The scaling exponent a can be calculated as

$$a = \ln \lambda_{\max}/2\ln(b). \tag{12}$$

The spectral dimension d_s equals d_f/a. Substituting (2) into (12), we obtain

$$d_s = 2\ln\left(2(n+1) + \sum_{i=1}^{p-1}(m_i+1)\right)(\ln \lambda_{\max})^{1/2}. \tag{13}$$

If we change the scaling unit b and fix n, p and all $\{m_i\}$, we will have many different fractals with the same topology and different fractal dimensions. However, the spectral dimension does not change at all because d_s is independent of b and dependent on $\lambda_{\max}$ which is only a function of n, p and $\{m_i\}$. In general, if we have a Koch curve with w kinds of branching points, we can also obtain w renormalised equations of $\{\tilde{\alpha}_i\colon i = 1, w\}$, then expand them near zero $\{\alpha_i\}$ to obtain a set of linear renormalisation equations as (8). The $\lambda_{\max}$ can be obtained in principle and is not related to the self-similar scaling unit b so that Koch curves with the same topology must have the same spectral dimension.

The following cases have been calculated.

(i) $n = 2$, $m_1 = 4$, $\{m_i = m_{i-1} + 2\colon i = 2, \ldots, p-1\}$ and $p = 3$-200. The results show that d_f approaches infinity and d_s to the limit of 2 with increasing p.

(ii) $p = 15$, $\{m_i = m\colon i = 1, \ldots, p-1\}$, $m = 1, \ldots, 200$ and $n = 1, \ldots, 10$. When p and n are fixed and m is changed, the spectral dimensions will be changed as well as the fractal dimensions. It has been found that the spectral dimension d_s has a maximum value at some m. We also calculated the ratio β $(=\alpha_2^*/\alpha_p^*)$ of the eigenvector of equation (10) related to $\lambda_{\max}$ and found that it exactly equals $2/p$ for each case.

For the infinite cluster at critical percolation, a random fractal, Alexander and Orbach (1982) found a weak dependence of spectral dimensionality d_s on the Euclidean dimensionality d as well as the fractal dimensionality d_f, and conjectured that $d_s = \frac{4}{3}$ for any d. Here we can find that d_s has some dependence on d_f and approaches two as $d_f \to \infty$. One can conjecture that the dependence of d_s on d_f may be different for deterministic fractals (such as the Koch curve, the Sierpinski gasket, etc) and infinite clusters at percolation threshold.

The Einstein relation (1) can be rewritten as (Rammal and Toulouse 1983)

$$2a - d_f = \tilde{t}. \tag{14}$$

The scaling expont t of DC conductivity can be found as

$$\tilde{t} = \ln\left[2(n+1)+1\left(\sum_j (1/m_j)\right)^{-1}\right](\ln b)^{-1}. \tag{15}$$

We have calculated $2a - d_f$ and $\tilde{t}$ for any case and our results show that they are exactly equal. Therefore, we can obtain an analytical formula for the spectral dimension d_s

$$d_s = 2d_f/(\tilde{t} + d_f) \tag{16}$$

where d_f and $\tilde{t}$ are defined by (3) and (15). From (16), (3) and (15), we can find that d_f and d_s may become equal and approach unity only in the limiting case of $n \to \infty$. When $p \to \infty$ or $m_i \to \infty$, d_f will become infinitive and d_s will approach two. It results that the maximum d_s is two for such a kind of Koch curve.

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